Skolem functions and Hilbert's ϵ-terms in Free Variable Tableau Systems

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Outline

- Motivation
- Skolemization based tableau calculus
- The δ^{ϵ}-rule..
- ..and its Skolemization based counterpart: the $\delta^{\mathbf{sk}}$-rule
- Identification of the δ^{ϵ}- and the $\delta^{\mathbf{sk}}$-rule
- Conclusions

Motivation

Existential quantifiers elimination in tableau proofs is dealt with by a tableau expansion rule

Approach

Existential quantifiers are replaced by suitable terms during the proof construction

Construction of instantiation terms is generally based on Skolemization: the initial language is enriched with Skolem terms

Skolem terms may be assimilated to the terms of the language → easily manipulated in deductions

Substitutivity: Infer $Q(a)$ from $(\forall x)Q(x)$

Motivation (ctnd.)

Hilbert's ϵ-symbol is an operator for the formation of terms that replace quantifiers in standard predicative logic

$\epsilon x.\varphi$: *an element of the domain satisfying* φ *if such an element does exist, an arbitrary element of the domain otherwise*

Applications: foundational issues, philosophy, non classical logics...

Giese and Ahrendt (1999) investigate how to apply ϵ-terms in automated deduction as syntactical objects

- Efficiency (comparisons with Skolemization based rules)
- No enriched signature

Our work

Mapping the δ^{ϵ}-rule in the context of standard Skolemization based δ-rules

How?

- Construct the $\delta^{\mathbf{sk}}$-rule (based on Davis, Fechter 1991)
- Prove soundness of the $\delta^{\mathbf{sk}}$-rule by applying the generic sound Skolemization based δ-rule (Cantone, Nicolosi Asmundo 2006)
- identify δ^{ϵ}- and $\delta^{\mathbf{sk}}$-rule in the context of the same tableau calculus (isomorphism of the tableau proofs)

As a result, the ϵ-based rule is endowed with

- a mechanism to construct canonical models
- an alternative soundness proof

A generic free variable tableau calculus

Smullyan's uniform notation

Expansion rules

$$\frac{\alpha}{\begin{array}{c}\alpha_1\\ \alpha_2\end{array}} \qquad \frac{\beta}{\beta_1 \mid \beta_2} \qquad \frac{\gamma}{\gamma_0(x)} \qquad \frac{\delta}{\delta_0(f(\overrightarrow{S}))}$$

Closure

construct a substitution σ such that for every branch θ, there exist two literals ψ, ψ' such that $\psi\sigma = \mathsf{C}(\psi')\sigma$

Skolemization based δ-rules

The δ-expansion rule has undergone several optimizations

- δ-rule (M. Fitting, 1990)
- δ^{+}-rule (R. Hähnle and P. Schmitt, 1994)
- $\delta^{+^{+}}$-rule (B. Beckert, R. Hähnle and P. Schmitt, 1993)
- δ^{*}-rule (M. Baaz and C. Fermüller, 1995)
- $\delta^{*^{*}}$-rule (D. Cantone and M. Nicolosi Asmundo, 2000)

Efficiency

▷ Reducing the number of new Skolem function symbols

▷ Using less free variables as arguments of the Skolem term

⟶ easier to close branches ⟶ shorter proofs

Examples of Skolem terms construction

$$\delta = (\exists x)(P(x, g(x_1)) \wedge Q(x_2, x_3))$$

$P(f(x_1, x_2, x_3, \_\_), g(x_1)) \wedge Q(x_2, x_3)$ δ-rule

$P(f(x_1, x_2, x_3), g(x_1)) \wedge Q(x_2, x_3)$ δ^{+}-rule

$P(f_{[\delta]}(x_1, x_2, x_3), g(x_1)) \wedge Q(x_2, x_3)$ $\delta^{+^{+}}$-rule

$P(f_{[\delta]}(x_1), g(x_1)) \wedge Q(x_2, x_3)$ δ^{*}-rule

$P(f_{[\varphi]}(g(x_1)), g(x_1)) \wedge Q(x_2, x_3)$ $\delta^{*^{*}}$-rule, $\varphi = P(\mathsf{x}_0, \mathsf{x}_1)$

$\sigma = \{\mathsf{x}_0/x, \mathsf{x}_1/g(x_1)\}$

A generic δ-rule

$$\frac{\delta}{\delta_0(f(\overrightarrow{S}))}$$

- f is a Skolem function symbol new to δ
- $\overrightarrow{S}$ is an ordered tuple of terms
- $f(\overrightarrow{S})$ is computed by a function $S_{\boldsymbol{\delta}}(\mathcal{T}, m, n)$
  - $\mathcal{T}$ is the current tableau
  - m is the index of the branch to be expanded
  - n is the position of the δ-formula to be instantiated.

Skolemization

Let δ be an existentially quantified formula, we skolemize δ by replacing the existential quantifier with a Skolem term $f(x_1, \ldots, x_n)$ where

- f is new to δ
- $(x_1, \ldots, x_n)$ contains all the free variables in δ

Skolemization preserves satisfiability

δ-rules and Skolemization

Which formulae does a δ-rule variant skolemize?

- The δ-, δ^{+} and $\delta^{+^{+}}$-rules skolemize every formula in the proof
- The δ^{*}-rule skolemizes

$$\delta_1 = (\exists x)P(x, g(x_1)) \longrightarrow P(f_{[\delta_1]}(x_1), g(x_1))$$

- The $\delta^{*^{*}}$-rule skolemizes

$$\delta_2 = (\exists x_0)P(\mathsf{x}_0, \mathsf{x}_1) \longrightarrow P(f_{[\delta_2]}(\mathsf{x}_1), \mathsf{x}_1)$$

A generic Skolemization based sound δ-rule

Central Idea

$$\delta \xrightarrow{\text{sat. pres. trans.}} \delta^s \rightarrow \boxed{\text{Skolemization}} \rightarrow \delta_0^s(f(\overrightarrow{H})) \xrightarrow{\text{sat. pres. trans.}} \delta_0(f(\overrightarrow{S}))$$

Only formulae δ^s identical up to variable renaming may share the same function symbol

Main Features

The generic Skolemization based sound δ-rule can be used

- As a general schema for proving soundness of δ-rule variants
- To compare different δ-rule versions
- As a useful tool to detect unsound δ-rules

Formally defining Skolemization based sound δ-rules 1.

Condition 1.: Objects that a sound Skolemization based δ-rule has to provide

a. – A collection of Skolemizable δ-formulae Δ^s,

- For each δ^s, a nonempty set of function symbols, $\mathbf{sko}_{[\delta^s]}$
  - $\mathbf{sko}_{[\delta^s]}$ is used to skolemize $[\delta^s]$ only
  - Elements of $\mathbf{sko}_{[\delta^s]}$ are interpreted in a canonical way

Examples of Δ^s

- δ^*-rule: Δ^s is the collection of the relevant extracted formulae of the language
  - $\delta_1 = (\exists x)(P(x, x_1) \wedge Q(x, g(x_2))) \in \Delta^s \quad \longrightarrow \quad \mathbf{sko}_{[\delta_1]}$
  - $\delta_2 = (\exists x)(P(x, x_1) \wedge Q(x_2, g(x_3))) \notin \Delta^s$

- δ^{*^*}-rule: Δ^s is the collection of the relevant extracted key formulae of the language
  - $\delta_1 = (\exists x)(P(x, x_1) \wedge Q(x, g(x_2))) \notin \Delta^s$
  - $\delta_3 = (\exists \mathsf{x}_0)(P(\mathsf{x}_0, \mathsf{x}_1) \wedge Q(\mathsf{x}_0, \mathsf{x}_2)) \in \Delta^s \quad \longrightarrow \quad \mathbf{sko}_{[\delta_3]}$

Formally defining Skolemization based sound δ-rules 2.

Condition 1.: Objects that a sound **δ**-rule has to provide

b. For each δ individuated by $(\mathcal{T}, m, n)$, the **δ**-rule provides the corresponding

- abstraction formula δ^a, and substitution σ
- Skolemization formula δ^s and transformation formula $\xi = \xi(\delta^s)$
- Skolem term $f(\overrightarrow{H})$, $f \in \mathbf{sko}_{[\delta^s]}$ and $\overrightarrow{H}$ containing all the free variables in δ^s

Example with the δ^{*^*}-rule

$$\delta_1 = (\exists x)(P(x, x_1) \lor Q(x_2, g(x_3)))$$

- $\delta_1^a = (\exists \mathsf{x}_0)(P(\mathsf{x}_0, \mathsf{x}_1) \lor Q(\mathsf{x}_2, \mathsf{x}_3))$,
  $\sigma = \{\mathsf{x}_0/x, \mathsf{x}_1/x_1, \mathsf{x}_2/x_2, \mathsf{x}_3/g(x_3)\}$
- $\delta_1^s = (\exists \mathsf{x}_0)P(\mathsf{x}_0, \mathsf{x}_1)$
- $\xi_1(\delta_1^s) = (\exists \mathsf{x}_0)P(\mathsf{x}_0, \mathsf{x}_1) \lor Q(\mathsf{x}_2, \mathsf{x}_3)$
- $\mathbf{sko}_{[\delta^s]} = \{f\}$, $\overrightarrow{H} = (\mathsf{x}_1) \longrightarrow f(\overrightarrow{H}) = f(\mathsf{x}_1)$

Formally defining Skolemization based sound δ-rules 3.

Conditions 2-4: relationships that have to hold between the objects

Condition 2 σ free for δ^a, $\delta^a_0(f(\overrightarrow{H}))$, ξ, and $\xi(\delta^s_0(f(\overrightarrow{H})))$

Condition 3 δ^s subformula of ξ, occurring *only* positively in it

Condition 4 the quantified variable of δ^s occurs in ξ only in occurrences of δ^s

Formally defining Skolemization based sound δ-rules 4.

Conditions 5-8: implications that have to hold between the objects

Condition 5 $\models \delta \supset \delta^a \sigma$

Condition 6 $\models \delta^a_0(x) \supset \xi(\delta^s_0(x))$

Condition 7 $\models \delta^a \supset (\forall x)(\xi(\delta^s_0(x)) \supset \delta^a_0(x))$

Condition 8 $\models (\delta^a_0(f(\overrightarrow{H})))\sigma \supset \delta_0(f(\overrightarrow{H})\sigma)$.

Giese and Ahrendt's δ^{ϵ}-rule

$$\frac{\delta}{\delta_0(\epsilon x.\delta_0(x))}$$

- x quantified variable in δ
- $\epsilon x.\delta_0(x)$ ϵ-term associated to δ

Example

$$\begin{aligned} \delta &= (\exists x)P(x, f(y)) \\ \epsilon x.\delta_0(x) &= \epsilon x.P(x, f(y)) \\ \delta_0(\epsilon x.\delta_0(x)) &= P(\epsilon x.P(x, f(y)), f(y)) \end{aligned}$$

Interpretation of ϵ-terms

First order structures are endowed with a function $\boldsymbol{\epsilon\text{-}val}$ mapping any ϵ-term and variable assignment into an element of the domain.

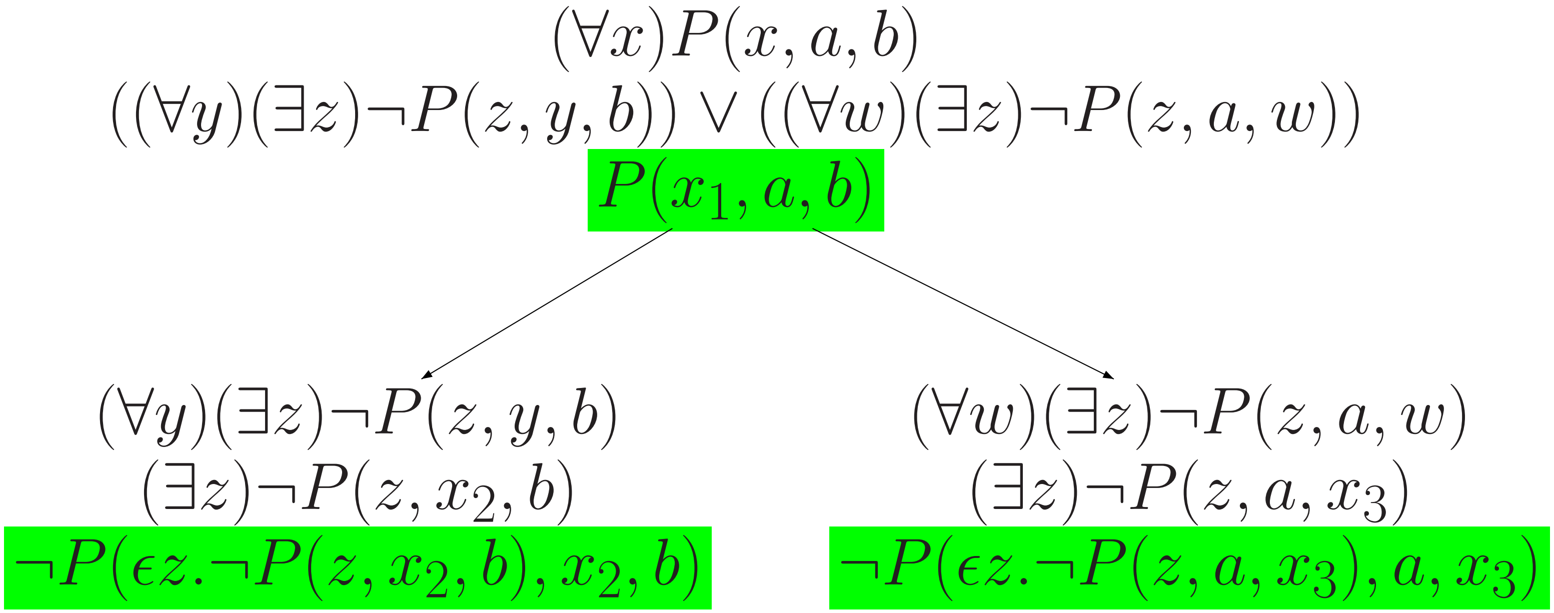
Substitutive structures

- the evaluation of the ϵ-term should depend only on the evaluation of the variables occurring free in it
- Substitutivity property:

  To deduce $Q(\epsilon y.P(a, y))$ from $(\forall x)Q(\epsilon y.P(x, y))$

- an ϵ-term $\epsilon x.\varphi$ should denote an element of the domain that, assigned to the variable x, satisfies φ

Example with the δ^{ϵ}-rule



$$(\forall x)P(x,a,b)$$

$$((\forall y)(\exists z)\neg P(z,y,b)) \vee ((\forall w)(\exists z)\neg P(z,a,w))$$

$$P(x_1,a,b)$$

$$(\forall y)(\exists z)\neg P(z,y,b)$$

$$(\exists z)\neg P(z,x_2,b)$$

$$\neg P(\epsilon z.\neg P(z,x_2,b),x_2,b)$$

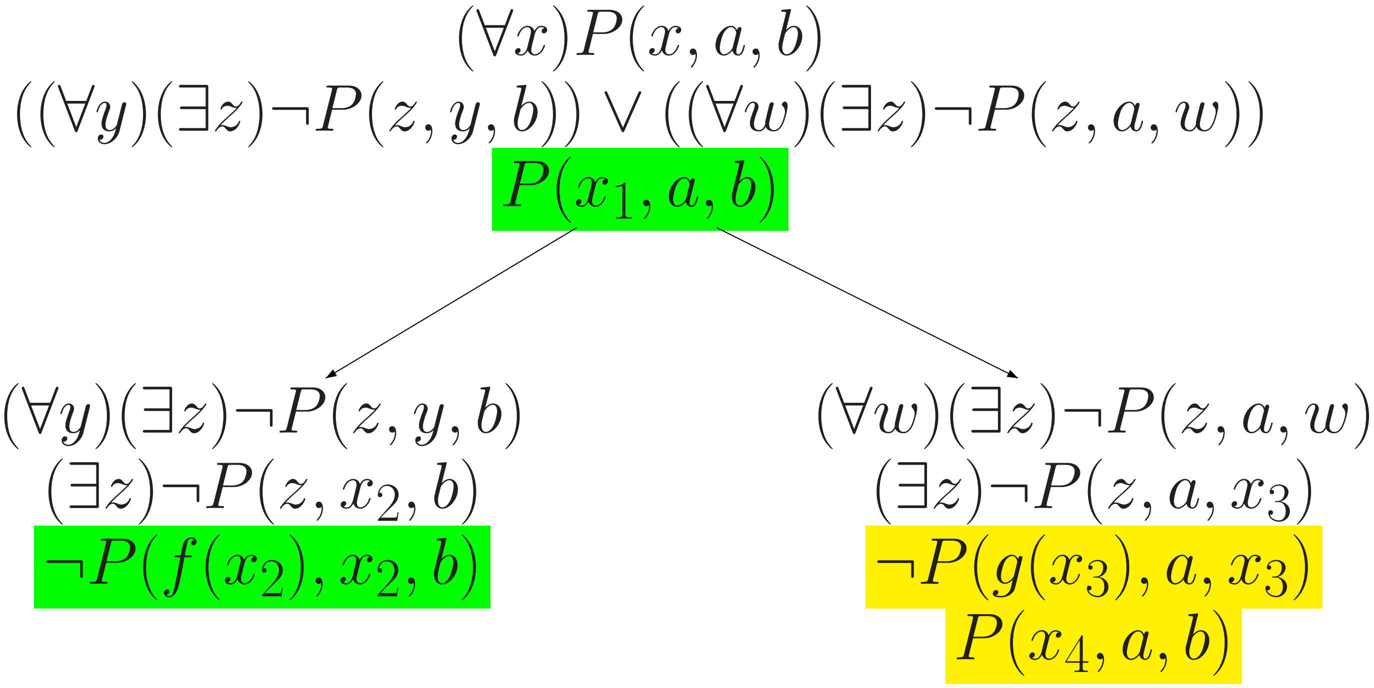
$$(\forall w)(\exists z)\neg P(z,a,w)$$

$$(\exists z)\neg P(z,a,x_3)$$

$$\neg P(\epsilon z.\neg P(z,a,x_3),a,x_3)$$

$$\sigma = \{x_1/\epsilon z.\neg P(z,a,b), x_2/a, x_3/b\}$$

Example with other δ-rules



$$(\forall x)P(x,a,b)$$

$$((\forall y)(\exists z)\neg P(z,y,b)) \vee ((\forall w)(\exists z)\neg P(z,a,w))$$

$$P(x_1,a,b)$$

$$(\forall y)(\exists z)\neg P(z,y,b)$$

$$(\exists z)\neg P(z,x_2,b)$$

$$\neg P(f(x_2),x_2,b)$$

$$(\forall w)(\exists z)\neg P(z,a,w)$$

$$(\exists z)\neg P(z,a,x_3)$$

$$\neg P(g(x_3),a,x_3)$$

$$P(x_4,a,b)$$

$$\sigma = \{x_1/f(a), x_2/a, x_4/g(b), x_3/b\}$$

Quasi-key Formulae

Quasi-key Formulae (based on Davis, Fechter 1991)

Represent the formula classes to which Skolem function symbols are assigned

Each formula of the language corresponds to a unique quasi-key formula (of which it inherits the related Skolem function symbol)

Example

$$\begin{tikzcd}
\boxed{(\exists x)P(x, y)} \arrow[dr, "\{\mathsf{x}_1/y\}"] & \boxed{(\exists x)P(x, f(f(z)))} \arrow[d, "\{\mathsf{x}_1/f(f(z))\}"] & \boxed{(\exists x)P(x, h(g(z), x))} \arrow[dl, "\{\mathsf{x}_1/h(g(z), x)\}"] \\
& \boxed{P(x, \mathsf{x}_1)} &
\end{tikzcd}$$

The δ^{sk}-rule

$$
\begin{aligned}
\delta &= (\exists z)\neg P(z, x_2, a) \\
QKey(\delta) &= \neg P(z, \mathsf{x}_1, \mathsf{x}_2) \\
\sigma &= \{\mathsf{x}_1/x_2, \mathsf{x}_2/a\} \\
\delta_0(f(\overrightarrow{S})) &= \neg P(f(\mathsf{x}_1, \mathsf{x}_2), \mathsf{x}_1, \mathsf{x}_2)\sigma \\
&= \neg P(f(x_2, a), x_2, a)
\end{aligned}
$$

Instantiating the generic δ-rule to the δ^{sk}-rule

- Δ^s is the collection of all $\boldsymbol{\delta}$-formulae of the language $\delta^s = (\exists x) QKey(\delta_0(x), x)$

- $\mathbf{sko}_{[\delta^s]}$, a set containing one function symbol for each quasi-key formula of $[\delta^s]$
- Given a $\boldsymbol{\delta}$-formula δ occurring on a branch θ of a tableau $\mathcal{T}$, we put:
  - $\delta^a = \delta^s = \xi = (\exists x) QKey(\delta_0(x), x)$,
  - σ, $\delta = \delta^a \sigma$;
  - $\overrightarrow{H}$ is a tuple containing all the free variables in δ^s, ordered as they appear on δ^s from left to right, while f is the function symbol in $\mathbf{sko}_{[\delta^s]}$ associated to δ^s.

Isomorphism between tableau proofs with the $\delta^{\mathbf{sk}}$-rule and the δ^{ϵ}-rule

Intuition

The application of either the $\delta^{\mathbf{sk}}$-rule or the δ^{ϵ}-rule gives rise to identical proofs up to the kind of terms (either Skolem terms or ϵ-terms) used to expand existentially quantified formulae

Ingredients

(a) a bijection from $\mathcal{L}^{+}_{\Sigma_{\mathbf{sko}}}$ to $\mathcal{L}^{+}_{\Sigma_{\epsilon}}$

(b) a mechanism to construct substitutive canonical structures for $\mathcal{L}^{+}_{\Sigma_{\epsilon}}$

Quasi-key formulae in $\mathcal{L}^{+}_{\Sigma_{\epsilon}}$

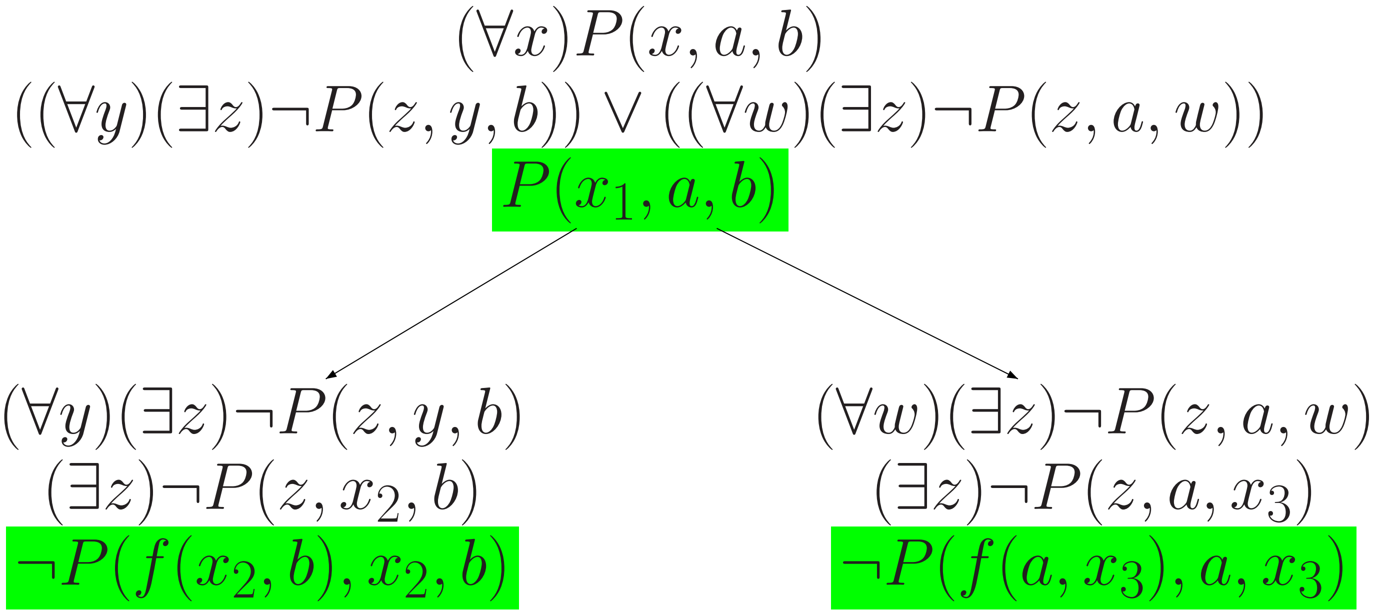
Example

Identical proofs up to the terms used to instantiate existentially quantified formulae

(1.) $P(x_1, x_2)$
(2.) $(\forall z)(\exists x)\neg P(x, g(z))$
(3.) $(\exists x)\neg P(x, g(x_3))$
(4.) $\neg P(f(g(x_3)), g(x_3))$

(1.) $P(x_1, x_2)$
(2.) $(\forall z)(\exists x)\neg P(x, g(z))$
(3.) $(\exists x)\neg P(x, g(x_3))$
(4.) $\neg P(\epsilon x.\neg P(x, g(x_3))), g(x_3))$

Example with the δ^{sk}-rule



$$(\forall x)P(x,a,b)$$

$$((\forall y)(\exists z)\neg P(z,y,b)) \vee ((\forall w)(\exists z)\neg P(z,a,w))$$

$$P(x_1,a,b)$$

$$(\forall y)(\exists z)\neg P(z,y,b)$$

$$(\exists z)\neg P(z,x_2,b)$$

$$\neg P(f(x_2,b),x_2,b)$$

$$(\forall w)(\exists z)\neg P(z,a,w)$$

$$(\exists z)\neg P(z,a,x_3)$$

$$\neg P(f(a,x_3),a,x_3)$$

$$\sigma = \{x_1/f(a,b), x_2/a, x_3/b\}$$

Conclusions

- Skolem terms constructed by the $\delta^{\mathbf{sk}}$-rule, and ϵ-terms produced by the δ^{ϵ}-rule can be considered essentially the same thing

- Conjuncture of two tendencies:

1. Construction of Skolem terms reflecting the meaning of the instantiation
2. Treating ϵ-terms as syntactical objects suitable for deduction

Future work

- For the ϵ-based approach: to individuate a generic ϵ-based sound δ-rule
- For the Skolemization based approach: investigate on extensional semantics